Introduction to Networks

6/8/2022

Agenda:
- Finish going over worksheet from yesterday
- Intro to networks

From worksheet due today (worksheet 2)

- Problem 3: Compute B^2 when $B = \begin{bmatrix} 0 & -1 & -2 \\ 1 & -10 & 3 \\ 1 & 0 & -1 \end{bmatrix}$

solution:

$$B^2 = \begin{bmatrix} 0 & -1 & -2 \\ 1 & -10 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 1 & -10 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-1-2 & 0+10+0 & -1 \\ 0-10+3 & 99 & -35 \\ 0+0-1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 10 & -1 \\ -7 & 99 & -35 \\ -1 & -1 & -1 \end{bmatrix}$$

- Problem 5: eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$0 = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(4-\lambda) - 2\cdot 2$$

$$= 4 - \lambda - 4\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 5\lambda$$

$$= \lambda(\lambda - 5)$$

eigenvalues: $\lambda = 0, 5$

eigenvector for $\lambda = 0$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - \lambda I)v$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\Rightarrow$ $0 = 1v_1 + 2v_2$

$v_1 = -2v_2$

set $v_2 = 1$

$\Rightarrow$ $v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

eigenvector for $\lambda = 5$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$0 = -4v_1 + 2v_2$$
$$v_1 = \frac{1}{2} v_2$$

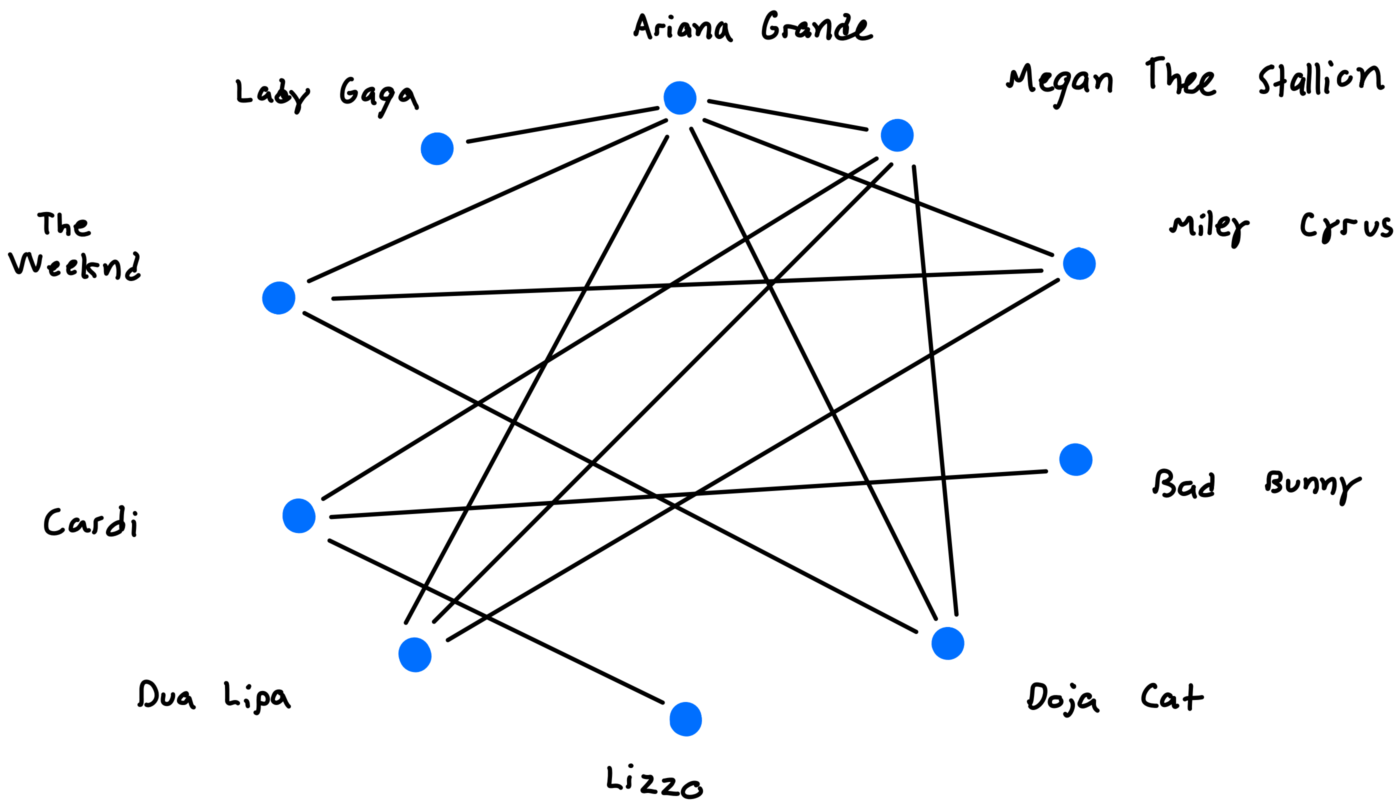
$$v = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Introduction to Networks

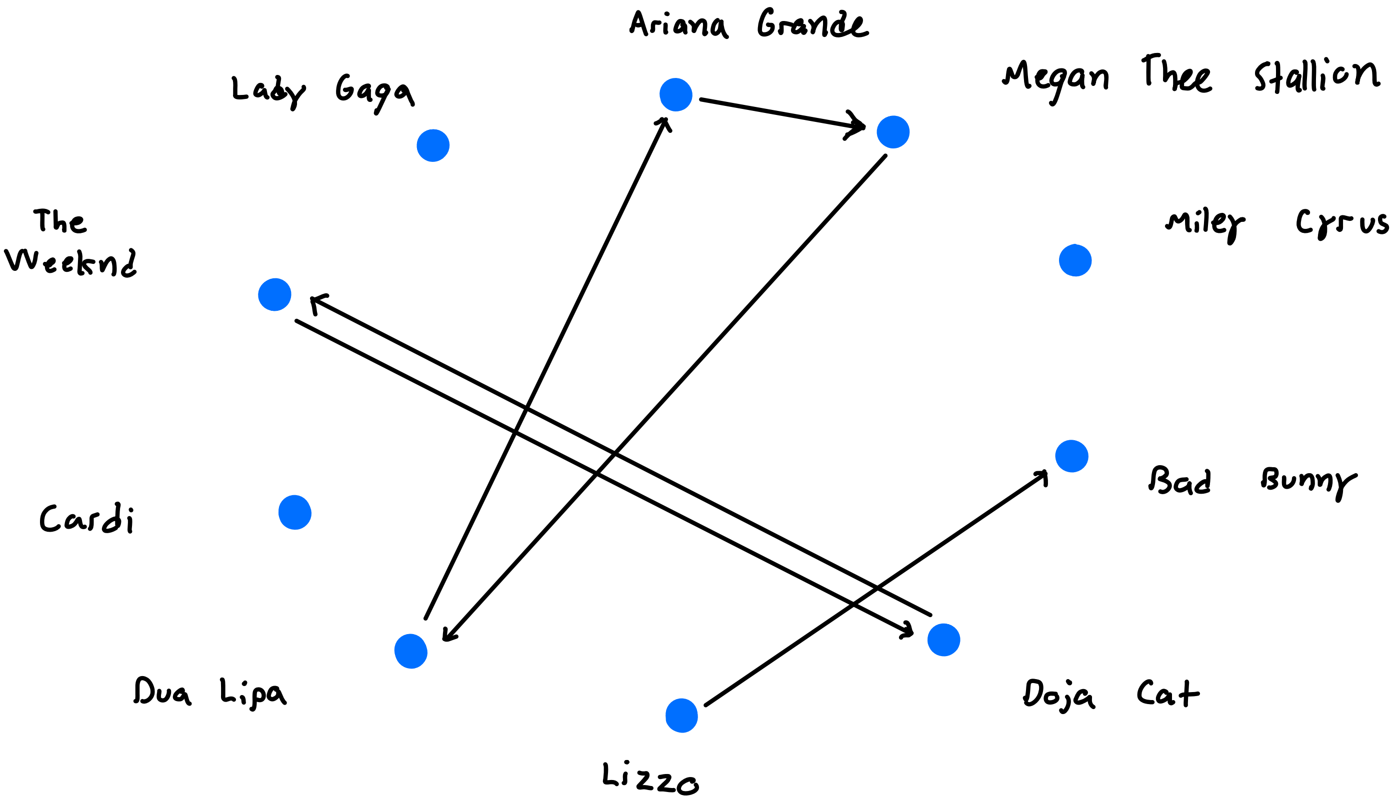
Recall: a network is a set of nodes connected by edges. Nodes usually correspond to physical objects (e.g. people, animals, street corners). Edges correspond to connections between nodes (e.g. friends on any social media platform, food web).

Alternative terminology: network = graph
node = vertex
edge = connection

ex) Music collabs



ex) Who likes whose music?



These two networks have the same nodes but completely different edges. This manifests the need to put networks into 2 categories: directed and undirected.

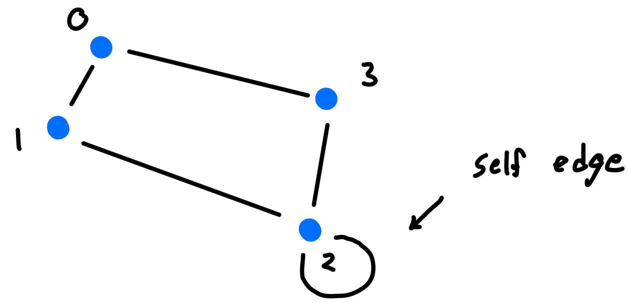
An undirected network has edges that do NOT have direction.
A directed network has edges that do have direction.

The First example above would be undirected while the second one would be directed.

- Some important definitions

  - self loop: Edge in a network which starts and ends at the same node.

ex)



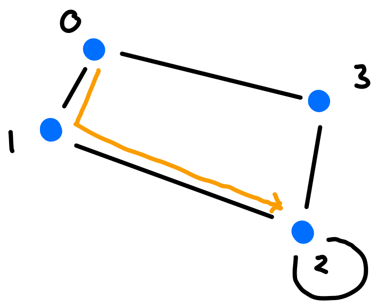
  - degree of a node: the number of connections that a node has to any node in the graph.

ex) The degree of node 1 above is 2.
ex) The degree of node 2 above is 3.

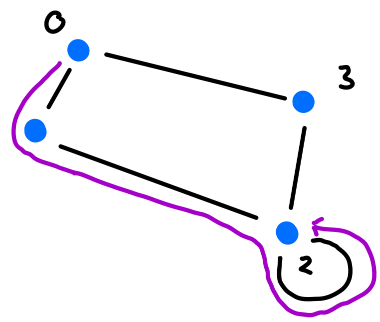
- Any two connected nodes are said to be adjacent.

  ex) Nodes 1 and 2 above are adjacent.
  ex) Nodes 1 and 3 above are not adjacent.

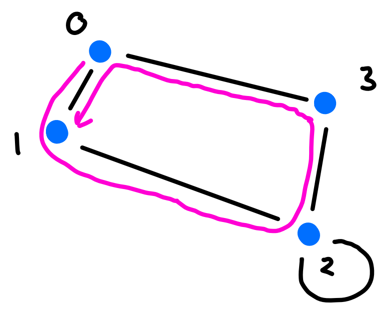
- A path is a sequence of edges and nodes. A path is simple if each edge in the sequence only appears once. A cycle is a path whose initial and final nodes are the same. The length of a path is equal to the number of edges in the path.



path
also simple path
not a cycle
length = 2



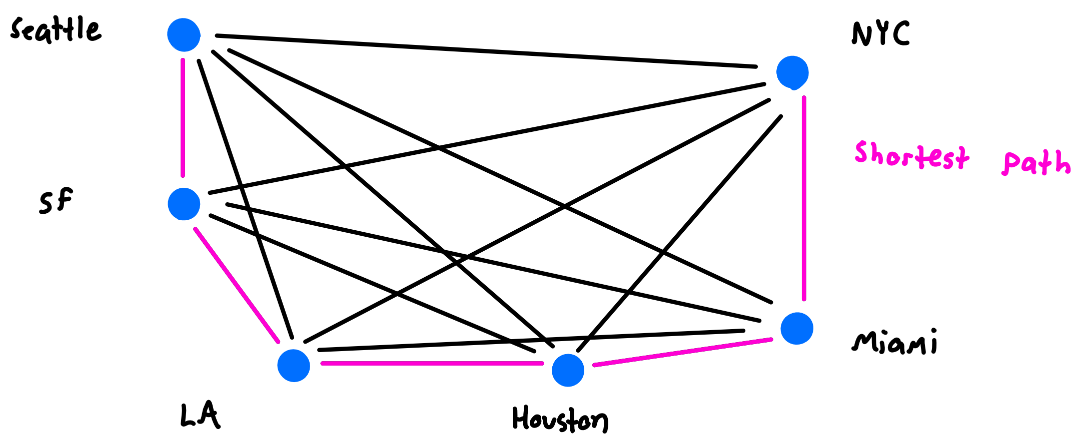
simple : yes
cycle : no
length : 3



simple : no
cycle : no
length : 5

With this we can start asking questions like " what is the shortest that goes through all nodes in a graph? "

ex)



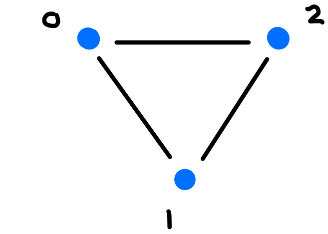
- There will be three important matrices to help us desribe graphs.

  1) Adjacency matrix: way to represent a network as something that a computer can understand.

  For a graph with n nodes, the adjacency matrix will be an $n \times n$ matrix with the following entries:

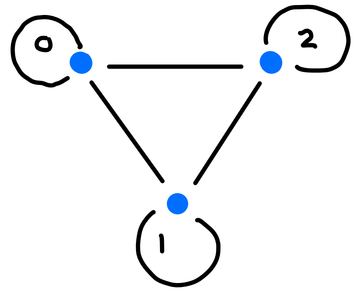
$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from node } i \text{ to node } j \\ 0 & \text{if there is no edge from node } i \text{ to node } j \end{cases}$$

ex)



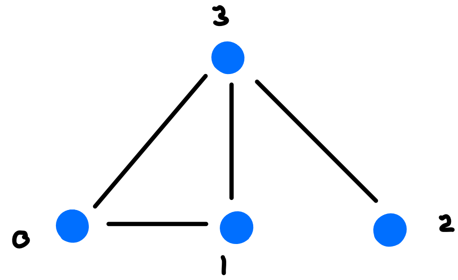
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

ex)



$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

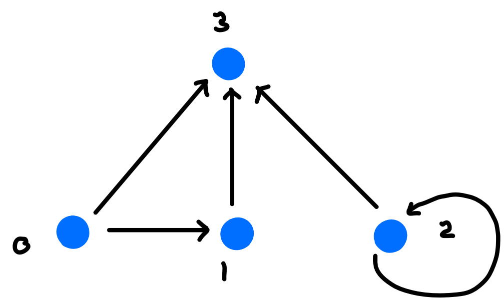
ex)



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

★ the i^{th} row of A corresponds to edges from the i^{th} node.
For example, the 0^{th} row of A corresponds to edges from the 0^{th} node.

ex)



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$