Matrix Operations and Eigen-stuff

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Agenda:
- Finish matrix operations
- Solving systems of equations
- Eigenvectors and eigenvalues

Matrix operations continued

- Dot product: Multiply two vectors by multiply then summing entries
  Requirement to do dot product: vectors must have same size
  For 2D vectors
  $$V = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$
  the dot product is: $V \cdot U = 3 \cdot 5 + (-1) \cdot 2$
  $= 15 - 2$
  $= 13$

  For n-D vectors
  $$V = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}, \quad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$
  the dot product is: $V \cdot U = \sum_{i=0}^{n-1} v_i \cdot u_i$

  Compute $\begin{bmatrix} 10 \\ -1 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 2 \\ -1/3 \\ 1 \end{bmatrix}$

- Matrix-matrix product: multiply two matrices using the dot product
  Requirement: number of columns in left matrix must be the same as number of rows in right matrix.
  For example, Left matrix size: $n \times m$
  right matrix size: $m \times p$
  Result: matrix of size $n \times p$

Let's write out the equations:

$$L = \begin{bmatrix} L_{0,0} & L_{0,1} & \cdots & L_{0,m-1} \\ L_{1,0} & L_{1,1} & & \vdots \\ \vdots & & \ddots & \\ L_{n-1,0} & \cdots & & L_{n-1,m-1} \end{bmatrix}, \quad R = \begin{bmatrix} R_{0,0} & R_{0,1} & \cdots & R_{0,p-1} \\ R_{1,0} & R_{1,1} & & \vdots \\ \vdots & & \ddots & \vdots \\ R_{m-1,0} & \cdots & & R_{m-1,p-1} \end{bmatrix}$$

the matrix product $LR = C$ would be written as:

$$C = \begin{bmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,p-1} \\ C_{1,0} & C_{1,1} & & \vdots \\ \vdots & & & \\ C_{n-1,0} & & & C_{n-1,p-1} \end{bmatrix}$$

where $$C_{i,j} = \sum_{k=0}^{m-1} L_{i,k} \cdot R_{k,j} \qquad \text{for } i = 0, \ldots, n-1 \text{ and } j = 0, \ldots, p-1$$

i tells us the row of L to look at

j tells us the column of R to look at

For example, $C_{0,0} = L_{0,0} \cdot R_{0,0} + L_{0,1} \cdot R_{1,0} + \cdots + L_{0,m-1} \cdot R_{m-1,0}$

Write out $C_{0,1}$

ex) $$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -3 & 5 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 + 0 \cdot (-2) & -3 \cdot 1 + 0 \cdot 1 & 5 \cdot 1 + 0 \cdot 2 \\ 2 \cdot (-1) + (-1) \cdot (-2) & 2 \cdot (-3) - 1 \cdot 1 & 2 \cdot 5 + (-1) \cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -3 & 5 \\ 0 & -7 & 8 \end{bmatrix}$$

ex) $$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 1 \\ 0 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

- Determinant of a matrix (we will only focus on 2×2 matrices, for now)

  consider matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  then the determinant is $\det(A) = ad - bc$

ex) $$\det\left(\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}\right) = 1 \cdot (-1) - 5 \cdot 0$$
$$= -1$$

ex) $$\det\left(\begin{bmatrix} 1/2 & 1 \\ -1 & 2 \end{bmatrix}\right)$$

Eigenvalues and eigenvectors

Again we'll only focus on 2×2 matrices but if your project focuses on eigenvalues and "spectral" methods then you will need to know how to compute eigenvalues and eigenvectors for larger matrices.

Eigenvectors are special vectors associated with a matrix that help describe the geometry of a matrix. specifically they are vectors that are "stretched" by a matrix and the "stretching" factor is the eigenvalue.

The eigenvectors v_i and eigenvalues λ_i of matrix A satisfy

$$Av_i = \lambda_i v_i$$

steps for computing eigenvalues and eigenvectors:

1. Find solutions λ to $\det(A - \lambda I) = 0$
2. For each λ, solve the linear system
$$(A - \lambda I)v = \vec{0}$$
for the eigenvector v.

Note $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the "identity matrix"

$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the "zero vector"

ex) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Step 1:

$$\begin{aligned} 0 &= \det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) \\ &= (-\lambda)(-3-\lambda) - (1)(-2) \\ &= \lambda^2 + 3\lambda + 2 \\ 0 &= (\lambda+1)(\lambda+2) \end{aligned}$$

$\Rightarrow \lambda = -1, -2$

Step 2: For $\lambda = -1$, solve

$$(A - \lambda I)v = 0$$
$$(A + 1 \cdot I)v = 0$$
$$\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} v_1 + v_2 \\ -2v_1 - 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad v_1 + v_2 = 0$$
$$v_1 = -v_2$$

choose $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = -2$, solve

$$(A - \lambda I)v = 0$$
$$(A + 2I)v = 0$$
$$\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$
$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2v_1 + v_2 \\ -2v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad 2v_1 + v_2 = 0$$
$$v_2 = -2v_1$$

choose $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

thus the eigenvectors of A are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

with eigenvalues $-1, -2$

Make sure to check results!

Need to do the same for $\lambda = -2$ too!